J. Fluid Mech. (1980), *vol.* 100, *part* 1, *pp.* 31–34
Printed in Great Britain

Exact nonlinear helical oscillations in magnetohydrodynamics

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(Received 19 February 1980)

An exact solution of the nonlinear magnetohydrodynamic equations for a viscous incompressible fluid of finite conductivity is obtained, which represents a circularly polarized structure with time dependent amplitude in a uniform magnetic field. There is a phase difference of $\frac{1}{2}\pi$ between the spatial structures of the velocity and the magnetic field of the disturbance. In the inviscid perfectly conducting limit, this solution represents a standing helical oscillation which is a circularly polarized standing Alfvén wave of arbitrary amplitude, and the sum of the kinetic and magnetic energy densities of the oscillation are constant in the absence of any input of energy.

1. Introduction

Waves of arbitrary amplitude in a uniform incompressible inviscid fluid of infinite conductivity can propagate in either direction along a uniform magnetic field under the assumption that the sum of the total magnetic pressure and the hydrostatic pressure including the gravitational potential energy density is constant. These are known as Alfvén waves and propagate with the Alfvén velocity, which were found in 1944 by Wallén (Alfvén 1950). There is equipartition of energy between the magnetic energy density of the disturbance field and the kinetic energy density of the fluctuating fluid. In the present paper, an exact solution with non-propagating helical structure in a uniform magnetic field, and with time-dependent amplitude, is obtained.

2. Basic equations

When the displacement current in Maxwell's equations is neglected, the MHD equations for an incompressible fluid may be written by

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\nabla\left(\frac{P}{\rho}\right) + \frac{1}{\rho}(\mathbf{J}\times\mathbf{B}) + \nu\nabla^2\mathbf{u} + \mathbf{F} + \mathbf{G}, \tag{1}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{2}$$

$$\nabla\times\mathbf{B} = \mu\mathbf{J}, \tag{3}$$

$$\mathbf{J} = \sigma(\mathbf{E}+\mathbf{u}\times\mathbf{B}), \tag{4}$$

$$\nabla.\mathbf{B} = 0, \tag{5}$$

$$\nabla.\mathbf{u} = 0, \tag{6}$$

where $\mathbf{u}$ is the velocity, $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, $\mathbf{E}$ is the electric field, $\mathbf{F}$ is solenoidal part of external force acting upon a unit mass of the fluid, $\mathbf{G}$ is the acceleration due to gravity (equal to $-\nabla\psi$, where ψ is the gravitational potential), P is the pressure, ρ is constant mass density, μ is the permeability of free space, ν is constant kinematic viscosity, and σ is constant conductivity.

Let us write

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{b}, \tag{7}$$

so that $\mathbf{b}$ represents the field associated with the disturbance in a uniform magnetic field $\mathbf{B}_0$, and also

$$\mathbf{h}_0 = \mathbf{B}_0/(\mu\rho)^{\frac{1}{2}} \quad \text{and} \quad \mathbf{h} = \mathbf{b}/(\mu\rho)^{\frac{1}{2}}, \tag{8}$$

where $\mathbf{h}_0$ is the Alfvén velocity and its absolute value is denoted by h_0 below. When $\mathbf{u}$ and $\mathbf{h}$ are perpendicular to $\mathbf{h}_0$, we rewrite equations (1)–(6) in the form

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla\left(\frac{P}{\rho} + \psi + \tfrac{1}{2}\mathbf{u}^2\right) + \mathbf{u}\times(\nabla\times\mathbf{u}) + (\nabla\times\mathbf{h})\times\mathbf{h} + (\mathbf{h}_0.\nabla)\,\mathbf{h} + \nu\nabla^2\mathbf{u} + \mathbf{F}, \tag{9}$$

$$\frac{\partial \mathbf{h}}{\partial t} = \nabla\times(\mathbf{u}\times\mathbf{h}) + (\mathbf{h}_0.\nabla)\,\mathbf{u} + \eta\nabla^2\mathbf{h}, \tag{10}$$

where the magnetic diffusivity is given by $\eta = 1/\sigma\mu$.

3. Solution

For simplicity, we take $\mathbf{B}_0$ to be in the z direction. We seek solution of equations (9) and (10) in the form

$$\mathbf{u} = u(t)\,(\cos kz,\ -\sin kz, 0), \tag{11}$$

$$\mathbf{h} = h(t)\,(\sin kz, \cos kz, 0), \tag{12}$$

for which

$$\nabla\times\mathbf{u} = k\mathbf{u}, \quad \nabla\times\mathbf{h} = k\mathbf{h} \quad \text{and} \quad \mathbf{u}\times\mathbf{h} = u(t)\,h(t)\,(0,0,1), \tag{13}$$

$$(\mathbf{h}_0.\nabla)\,\mathbf{u} = -kh_0 u(t)\,(\sin kz, \cos kz, 0), \tag{14}$$

and

$$(\mathbf{h}_0.\nabla)\,\mathbf{h} = kh_0 h(t)\,(\cos kz,\ -\sin kz, 0). \tag{15}$$

Then, equations (9) and (10) become simply under hydrostatic equilibrium

$$\frac{du(t)}{dt} = kh_0 h(t) - \nu k^2 u(t) + F(t), \tag{16}$$

$$\frac{dh(t)}{dt} = -kh_0 u(t) - \eta k^2 h(t), \tag{17}$$

where a special solenoidal body force is given by

$$\mathbf{F} = F(t)\,(\cos kz,\ -\sin kz, 0). \tag{18}$$

When $F(t) = 0$, under inviscid fluid of infinite conductivity the solutions for $u(t)$ and $h(t)$ are as follows;

$$u(t) = u_0 \cos(kh_0 t + \epsilon), \tag{19}$$

$$h(t) = -u_0 \sin(kh_0 t + \epsilon), \tag{20}$$

where u_0 and ϵ are constant. In the terminology of Moffatt (1969), the magnetic and kinetic helicity densities of the oscillation are estimated to be $b_0^2/2k$ and $u_0^2/2k$, respectively, where $b_0^2 = \mu\rho u_0^2$, and the average is with respect to t. According to (12), the lines of force of the perturbed field $\mathbf{B}_0 + \mathbf{b}$ are left-handed helices, and the sign of helicity is positive. Since the oscillation given by (11) and (12) is a helical, periodic and non-propagating mode, we may call it 'helical oscillation'. By changing the sign of x or y components in (11) and (12), we obtain right-hand helices, which also provide exact solution. Since the mathematical treatment is similar to the left-hand case, the right-hand one is not discussed. It may easily be shown that the helical oscillation is the standing counterpart of the circularly polarized Alfvén wave, which Moffatt (1978) calls a helicity wave, which is a well-known exact solution of the MHD equations.

To see the effects of viscous and ohmic diffusion, we derive equations for $u(t)$ and $h(t)$ from (16) and (17) as follows;

$$\frac{d^2u(t)}{dt^2} + k^2(\eta+\nu)\frac{du(t)}{dt} + k^2(h_0^2 + \eta\nu k^2)\,u(t) = \frac{dF(t)}{dt} + k^2\eta F(t), \tag{21}$$

$$\frac{d^2h(t)}{dt^2} + k^2(\eta+\nu)\frac{dh(t)}{dt} + k^2(h_0^2 + \eta\nu k^2)\,h(t) = -kh_0 F(t). \tag{22}$$

If $F(t) = 0$, both $u(t)$ and $h(t)$ satisfy same equation. Since these equations are now linear, the behaviour follows the well-known linear pattern, viz,

(i) when $h_0^2 \leqq (k^2/4)(\eta-\nu)^2$ where $\eta \neq \nu$, the solution is of damped type and not oscillatory in time, and

(ii) when $h_0^2 > (k^2/4)(\eta-\nu)^2$, the solution is of damped-type oscillation with frequency ω given by

$$\omega^2 = k^2h_0^2[1 - k^2(\eta-\nu)^2/4h_0^2], \tag{23}$$

and damping time constant τ given by

$$\tau = 2/k^2(\eta+\nu). \tag{24}$$

Under the special case of $\eta = \nu$, ω and τ are given by

$$\omega^2 = k^2h_0^2, \tag{25}$$

and

$$\tau = 1/(k^2\eta), \tag{26}$$

respectively. It is clear that magnetic and kinetic helicity are not conserved for these cases. When the function $F(t)$ is given, the forced solution may be obtained. But, since the force term (18) is necessarily artificial for the present, we do not discuss further.

The equation of energy conservation is given by (from (16) and (17))

$$\frac{d}{dt}\left(\tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho h^2\right) = -\rho k^2(\nu u^2 + \eta h^2) + \rho u(t)\,F(t), \tag{27}$$

where $u^2 = |\mathbf{u}|^2$ and $h^2 = |\mathbf{h}|^2$. The first term on the right represents an energy sink due to viscous and ohmic dissipation. The second term is energy source supplied by the external force per unit time. In the absence of viscosity and resistivity, the equation becomes

$$\tfrac{1}{2}\rho u^2 + \tfrac{1}{2}\rho h^2 = \text{const.} = \tfrac{1}{2}\rho u_0^2, \tag{28}$$

for $F(t) = 0$. It is found that the sum of the kinetic and magnetic energy density of the disturbance is always constant. Also, as is evident from (19) and (20), for the case of arbitrary amplitude Alfvén waves, there is equipartition of the kinetic and magnetic energy density at all times.

The author would like to acknowledge the valuable comment of the Editor and the encouragement of T. Namikawa and T. Miura.

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